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Path-Integral Approach to the Statistical Physics of One-Dimensional Random Systems

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A path-integral method is extended and developed to investigate the statistical physics of one-dimensional random systems. Evaluation of the one-particle partition function and density matrix is simplified to finding a solution for a second-order ordinary differential equation. This makes it possible to obtain analytic solutions or conduct accurate numerical calculations for the random systems. With this approach, an analytical solution for the Gaussian model is obtained and the statistical physics of the Frisch–Lloyd model is studied.

KEY WORDS: Path integral; partition function; random system; Gaussian model; Frisch–Lloyd model.

1. INTRODUCTION

Joaquin M. Luttinger was not only one of the major figures in statistical and condensed matter physics, but also a dedicated teacher and mentor. As his doctoral student, I still remember the days when we worked together at Columbia University. Joaquin became my advisor in early 1980. We met and discussed physics almost every day in the following two and half years. His enthusiasm, dedication, friendly spirit and careful teaching strongly influenced me. He distinguished himself in promoting the relationship between students and advisor, too. I miss him forever.

One project we worked together was a new method to obtain exact evaluation of Green's functions for a class of one-dimensional disordered systems. The method was a combination of the "replica trick" and the Feynman expression for the propagator in terms of path integral.[(1)] This

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method was later found very useful in study of the diffusion in random media.[2–4] The results for the diffusion are useful for polymer growth in a random environment,[5–7], chemical reaction with random nucleation centers, and biological multiplication with random nutrient concentration.[3]

In this paper, I am going to extend and further develop this method to investigate the statistical physics of one-dimensional random systems. The $1+1$-dimensional random potential was an important starting point in theoretical statistical physics and investigation of disordered systems. Recent experimental observation of mesoscopic vortex using micromechanical oscillators provided a direct test of these systems.[8] Such developments stimulate renewed interest in examining these random systems. As it will be shown later, our path-integral approach is a general and powerful tool in obtaining the partition function and density matrix for these systems.

As shown in Section 2, this path-integral approach enables me to derive a general analytical expressions for the partition function and density matrix for a class of one-dimensional random systems. These analytical results are new and general. They simplify the evaluation of the one-particle partition function and density matrix to finding a solution for a second-order ordinary differential equation. As ordinary differential equations are one of the most studied subjects, these results are useful and readily applicable. This makes it possible to obtain analytic solutions or conduct accurate numerical calculations for the random systems.

To illustrate the generality and usefulness of these analytical results, I apply the formalism to two popular random systems: In Section 3, I will derive the analytical solution for the Gaussian random system[9] and in Section 4 I will derive results for the Frisch–Lloyd random system.[10] While these two models were extensively studied, the main works were concentrated on the low energy tail, which corresponds to the low temperature limit in this paper. However, the analytical solution of the partition function and density matrices derived from the path-integral approach provides the results well beyond the low temperature limit. We have the partition function for the whole temperature range. These results show new features of the two models in the region where the low energy tail is not applicable.

2. DENSITY MATRIX AND CANONICAL PARTITION FUNCTION

Let us consider non-interacting particles in a random potential confined on a line of length L. The one-particle Hamiltonian is given by

$$H = \frac{p^2}{2m} + V(x) \tag{2.1}$$

where $V(x)$ is a random potential and $0 \leqslant x \leqslant L$. The un-normalized density matrix is given by

$$\rho = e^{-\beta H} \tag{2.2}$$

where $\beta = 1/k_B T$. I denote

$$\rho(x, x'; \beta) = \langle x | \, e^{-\beta H} \, | x' \rangle = \sum_n e^{-\beta E_n} \psi_n^*(x) \, \psi_n(x') \tag{2.3}$$

where ψ_n is the eigenfunction of H with eigenvalue E_n. The boundary condition is $\rho(x, x'; 0) = \delta(x - x')$. In this paper, $\langle \ \rangle$ is used for Dirac notations and $\langle \ \rangle_{\text{ave}}$ is denoted for the average over the random potential. Define

$$g(x, x'; E) = \frac{1}{\pi} \operatorname{Im} \langle x | \, \frac{1}{E - H - i\eta} \, | x' \rangle \tag{2.4}$$

Here the notations Re and Im are for the real part and imaginary part, respectively. It is clear that

$$\rho(x, x'; \beta) = \int_{-\infty}^{\infty} e^{-\beta E} g(x, x'; E) \, dE \tag{2.5}$$

Since the Hamiltonian H is real, for simplicity, we make all its eigenfunctions real. The above equation can be written into a path integral

$$g(x, x'; E) = -\frac{2}{\pi} \operatorname{Re} \frac{\int d\psi \, \psi(x) \, \psi(x') \exp[-i \int d\xi \, \psi(\xi)(E - H - i\eta) \, \psi(\xi)]}{\int d\psi \exp[-i \int d\xi \, \psi(\xi)(E - H - i\eta) \, \psi(\xi)]} \tag{2.6}$$

The replica trick enables us to write $g(x, x'; E)$ into

$$-\frac{2}{\pi} \operatorname{Re} \lim_{n \to 0} \int d\psi \, \psi_1(x) \, \psi_1(x') \times \exp\left[-i \int d\xi \, \psi \left(E + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \xi^2} \right) \psi + i \int V \psi^2 \, d\xi \right] \tag{2.7}$$

where ψ is an n-dimensional vector. As we take average over the random potential, only the term containing V needs to be considered. For a class of random potential, this average leads to

$$\left\langle \exp\left[i \int_0^L V(\xi) \, \psi^2(\xi) \, d\xi \right] \right\rangle_{\text{ave}} = \exp\left[-i \int_0^L d\xi \; U(\psi^2(\xi)) \right] \tag{2.8}$$

where the function U depends on individual models. In the following two sections we will derive this function for the Gaussian random potential and the Frisch–Lloyd random potential. In ref. 1, there were discussions about this sample average for other random systems.[1]

Introduce a transformation, $\psi \to \psi \sqrt{m/\hbar}$. The average $\langle g(x, x'; E)\rangle_{\text{ave}}$ is given by

$$-\frac{2m}{\hbar^2\pi}\,\text{Re}\lim_{n\to 0}\int d\psi\,\psi_1(x)\,\psi_1(x')$$
$$\times\exp\left[-i\int d\xi\left[\psi\left(\frac{mE}{\hbar^2}+\frac{1}{2}\frac{\partial^2}{\partial\xi^2}\right)\psi+U\left(\frac{\psi^2 m}{\hbar^2}\right)\right]\right] \tag{2.9}$$

As shown in ref. 1, the above equation is simplified to

$$\langle g(x, x'; E)\rangle_{\text{ave}} = -\frac{2m}{\hbar^2\pi}\,\text{Re}\int_0^\infty\int_0^\infty dr\,dr'\,\phi_0(r)\,\phi_0(r')\,G(r, r'; x-x') \tag{2.10}$$

where $G(r, r', t)$ is a Green's function satisfying the equation,

$$i\,\frac{\partial G(r, r', t)}{\partial t}=\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\right)+\frac{mE}{\hbar^2}r^2+U\left(\frac{r^2 m}{\hbar^2}\right)\right]G(r, r', t) \tag{2.11}$$

with the boundary condition $G(r, r'; 0)=r\delta(r-r')$; ϕ_0 is the solution of the equation

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{mE}{\hbar^2}r^2+U\left(\frac{r^2 m}{\hbar^2}\right)\right]\phi_0=0 \tag{2.12}$$

with the boundary condition $\phi_0(0)=1$ and $\phi_0(\infty)=0$. Using $u=r^2/2$, we transform Eq. (2.12) into the form

$$\frac{\partial^2\phi_0}{\partial u^2}-\left[\frac{2mE}{\hbar^2}+\frac{U(2um/\hbar^2)}{u}\right]\phi_0=0 \tag{2.13}$$

The sample-averaged density matrix is given by

$$\langle\rho(x, x'; \beta)\rangle_{\text{ave}}$$
$$=-\frac{2m}{\hbar^2\pi}\,\text{Re}\int_{-\infty}^{\infty}e^{-\beta E}\,dE\int_0^\infty\int_0^\infty dr\,dr'\,\phi_0(r)\,\phi_0(r')\,G(r, r'; x-x') \tag{2.14}$$

From Eq. (2.10), when $x = x'$, we have

$$\langle g(x, x; E)\rangle_{\text{ave}} = -\frac{2m}{\hbar^2 \pi} \operatorname{Re} \int_0^\infty \phi_0^2 \, du \tag{2.15}$$

Differentiating Eq. (2.13) with E, we get

$$\frac{\partial^2}{\partial u^2}\left(\frac{\partial \phi_0}{\partial E}\right) - \left[\frac{2mE}{\hbar^2} + \frac{U(2um/\hbar^2)}{u}\right]\left(\frac{\partial \phi_0}{\partial E}\right) - \frac{2m}{\hbar^2}\phi_0 = 0 \tag{2.16}$$

Multiplying Eq. (2.16) by ϕ_0, Eq. (2.13) by $\partial \phi_0/\partial E$, subtracting and integrating over u, we find

$$\frac{2m}{\hbar^2}\int_0^\infty \phi_0^2 \, du = \frac{\partial}{\partial E}\phi_0'(u)|_{u=0} \tag{2.17}$$

Hence we have

$$\langle g(x, x; E)\rangle_{\text{ave}} = -\frac{1}{\pi} \operatorname{Re}\left[\frac{\partial}{\partial E}\phi_0'(u)|_{u=0}\right] \tag{2.18}$$

The one-particle partition function f is then given by

$$\begin{aligned} f(\beta) &= \int \langle g(x, x; E)\rangle_{\text{ave}} \, e^{-\beta E} \, dx \, dE \\ &= -\frac{L\beta}{\pi}\int_{-\infty}^\infty dE \, e^{-\beta E} \operatorname{Re}[\phi_0'(u)|_{u=0}] \end{aligned} \tag{2.19}$$

The calculation of $f(\beta)$ is now simplified to finding $\phi_0'(u)_{|u=0}$ from Eq. (2.13). While the particular form of ϕ_0 depends the behavior of U, some general discussions are helpful. Let us denote two independent solutions of Eq. (2.13) as $\phi_0^{(1)}$ and $\phi_0^{(2)}$, satisfying $\phi_0^{(1)}(0) = 1$, $[d\phi_0^{(1)}/du]_{|u=0} = 0$, $\phi_0^{(2)}(0) = 0$, and $[d\phi_0^{(1)}/du]_{|u=0} = 1$, respectively. Since these two solutions have the boundary conditions at $u = 0$, they are easy to calculate. The solution ϕ_0 is a combination of $\phi_0^{(1)}$ and $\phi_0^{(2)}$,

$$\phi_0 = \phi_0^{(1)} + \mu \phi_0^{(2)} \tag{2.20}$$

The constant μ is determined by the condition

$$\mu = -\lim_{u \to \infty} \phi_0^{(1)}(u)/\phi_0^{(2)}(u) \tag{2.21}$$

Then, $\phi_0(\infty)=0$ and $\phi'_0(0)=\mu$. The partition function is given by

$$f(\beta)=-\frac{L\beta}{\pi}\int_{-\infty}^{\infty} dE\, e^{-\beta E}\,\mathrm{Re}(\mu) \tag{2.22}$$

The new analytical results in Eq. (2.19) and (2.22) are general and powerful. As the ordinary differential equations are one of the most studied subjects, these results are also practical and readily applicable. For many cases, such as for the Gaussian model, we have an analytical solution from Eq. (2.13). Then we have the analytical solution for the partition function. Even if an analytical solution for Eq. (2.13) is not available, such as for the Frisch–Lloyd model, Eq. (2.19) or Eq. (2.22) provides the direction for a straightforward and accurate numerical calculation.

In the following two subsequent sections, we will apply the above analytical results to find the partition function for the Gaussian model and Frisch–Lloyd model. These two applications are just an illustration, showing how we can readily apply the above analytical results to particular models for calculation.

3. GAUSSIAN MODEL

The white-noise Gaussian potential[9] is characterized by

$$\begin{aligned}\langle V(x_1)\cdots V(x_{2l-1})\rangle &= 0\\ \langle V(x_1)\cdots V(x_{2l})\rangle &= \sum_{i_1\cdots i_{2l}} \frac{D^n}{2^n}\,\delta(x_{i_1}-x_{i_1})\cdots\delta(x_{i_{2l-1}}-x_{i_{2l}})\end{aligned} \tag{3.1}$$

where $l=1, 2,...$ and $\sum_{i_1 ... i_{2l}}$ extends over all possible partitions of the $2l$ indices $(1, 2,..., 2l)$ into l pairs $(i_1, i_2)\cdots(i_{2l-1}, i_{2l})$. Hence, the average over the random potential in Eq. (2.8) leads to

$$\begin{aligned}&\left\langle \exp\left[i\int_0^L V(\xi)\,\psi^2(\xi)\,d\xi\right]\right\rangle_{\text{ave}}\\ &=\sum_{l=0}^{\infty}\frac{i^l}{l!}\int_0^L\cdots\int dx_1\cdots dx_l \langle V(x_1)\cdots V(x_l)\rangle\,\psi^2(x_1)\cdots\psi^2(x_l)\\ &=\sum_{m=0}^{\infty}\frac{i^{2m}}{(2m)!}\,\frac{D^m}{2^m}\,(2m-1)!\left[\int_0^L dx\,\psi^2(x)\right]^m\\ &=\exp\left[-\frac{D}{4}\int_0^L dx[\psi^2(x)]^2\right]\end{aligned} \tag{3.2}$$

Comparing with Eq. (2.8), we have $U(\psi^2) = -iD(\psi^2)^2/4$. Then from Eq. (2.13), we have the equation for ϕ_0,

$$\frac{\partial^2 \phi_0}{\partial u^2} + \left(-\frac{2mE}{\hbar^2} + i\,\frac{Dm^2}{\hbar^4}\,u \right) \phi_0 = 0 \tag{3.3}$$

which implies that ϕ_0 is an Airy function. After taking the boundary conditions into account, we represent ϕ_0 in the form

$$\phi_0 = \frac{(1 - (iDm/2E\hbar^2)\,u)^{1/2}\, H^{(1)}_{1/3}[2mD^{1/2}(-2E\hbar^2/Dm + iu)^{3/2}/(3i\hbar^2)]}{H^{(1)}_{1/3}[2^{5/2}\hbar(-E)^{3/2}/(3im^{1/2}D)]} \tag{3.4}$$

where $H^{(1)}_{1/3}$ is the Hankel function of the first kind. We then have

$$\phi_0'(u)_{|u=0} = -\frac{iDm}{4E\hbar^2} + \frac{(-2mE)^{1/2}\, H^{(1)\prime}_{1/3}[-2^{5/2}\hbar(E)^{3/2}/(3m^{1/2}D)]}{\hbar H^{(1)}_{1/3}[-2^{5/2}\hbar(E)^{3/2}/(3m^{1/2}D)]} \tag{3.5}$$

Introducing

$$Z(E) = (-E)^{1/2}\, H^{(1)}_{1/3}[-2^{5/2}\hbar(-E)^{3/2}/(3m^{1/2}D)] \tag{3.6}$$

we have

$$\phi_0'(u)_{|u=0} = -\frac{iDm}{2\hbar^2}\,\frac{Z'(E)}{Z(E)} \tag{3.7}$$

Hence

$$\langle g(x, x; E)\rangle_{\text{ave}} = -\frac{Dm}{2\pi\hbar^2}\,\text{Im}\,\frac{\partial}{\partial E}\left[\frac{Z'(E)}{Z(E)}\right] \tag{3.8}$$

The one-particle partition function f is given by

$$f(\beta) = \frac{-LDm\beta}{2\pi^2\hbar^2}\,\text{Im} \int_{-\infty}^{\infty} e^{-\beta E} \left[\frac{Z'(E)}{Z(E)}\right] dE \tag{3.9}$$

Before calculating Eq. (3.9), we note that $Z(E)$ satisfies the Airy equation

$$Z''(E) + \frac{8\hbar^2}{D^2 m}\,EZ(E) = 0 \tag{3.10}$$

The second independent solution of Eq. (3.10) is $Z^*(E)$, complex conjugate of $Z(E)$ for a real E. The Wrongskian of these two solutions is a constant, given by

$$Z'(E)\,Z^*(E) - Z(E)\,Z^{*\prime}(E) = -6i/\pi \tag{3.11}$$

Then from

$$\operatorname{Im}\left[\frac{Z'(E)}{Z(E)}\right] = -\frac{i}{2}\left[\frac{Z'(E)}{Z(E)} - \frac{Z^{*\prime}(E)}{Z^*(E)}\right] = \frac{-3}{\pi\,|Z(E)|^2} \tag{3.12}$$

we have

$$f(\beta) = \frac{3LDm\beta}{2\pi^3\hbar^2}\int_{-\infty}^{\infty}\frac{e^{-\beta E}}{|Z(E)|^2}\,dE \tag{3.13}$$

In calculating Eq. (3.13), we divide the integration into two parts $\int_{-\infty}^{0}$ and $\int_0^{\infty}$. After some algebra, we have

$$f(\beta) = \frac{3LDm\beta}{2\pi^3\hbar^2}\int_0^{\infty}\frac{d\eta}{\eta}\left[\frac{e^{\eta\beta/\beta_0}}{|H_{1/3}^{(1)}(-i\eta^{3/2})|^2} + \frac{e^{-\eta\beta/\beta_0}}{|H_{1/3}^{(1)}(-\eta^{3/2})|^2}\right] \tag{3.14}$$

where $\beta_0 = (2^{5/2}\hbar/3Dm^{1/2})^{2/3}$. Equation (3.14) is the analytical result for the partition of the Gaussian model. As $\eta \to \infty$, the asymptotic behavior for the two Hankel functions in Eq. (3.14) are different,

$$|H_{1/3}^{(1)}(-i\eta^{3/2})|^2 \to \frac{2}{\pi\eta^{3/2}}\exp(2\eta^{3/2}) \tag{3.15}$$

and

$$|H_{1/3}^{(1)}(-\eta^{3/2})|^2 \to \frac{2}{\pi\eta^{3/2}} \tag{3.15$'$}$$

The integral in Eq. (3.14) is convergent. In Fig. 1, we plot Log $f(\beta)$ versus β. It is noted that at low temperature, the behavior of $f(\beta)$ of the Gaussian model is completely different from the partition function of free particles. The reason is that the white-noise Gaussian potential is not bounded below. Although the probability is low to have a very large negative potential, it can happen during the sample average. At low temperature, some low energy states produced by such very negative potentials dominate the partition function. This changes the behavior of $f(\beta)$ and leads to a very large negative average energy at low temperature.

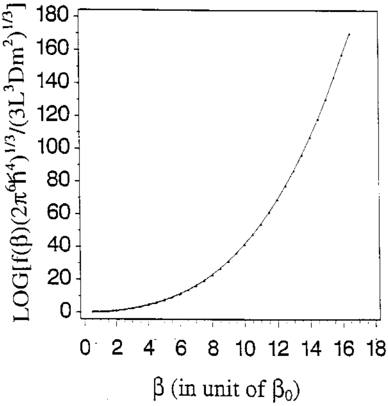

Fig. 1. Log $f(\beta)$ as a function of β/β_0 for the Gaussian model.

At high temperature, $\beta < \beta_0$, or $k_B T > (3Dm^{1/2}/2^{5/2}\hbar)^{2/3}$, the main contribution to $f(\beta)$ is from the second term in Eq. (3.14),

$$f(\beta) \to L\sqrt{\frac{m}{2\pi^3\hbar^2\beta}} \tag{3.16}$$

The average particle energy is $\varepsilon = k_B T/2$, a typical result in the high temperature limit.

At low temperature, $\beta > \beta_0$, or $k_B T < (3Dm^{1/2}/2^{5/2}\hbar)^{2/3}$, the main contribution to $f(\beta)$ is from the first term in Eq. (3.13). Using the saddle-point method, we have the asymptotic behavior,

$$f(\beta) \to \frac{LD^2 m^{3/2}\beta^{5/2}}{2^{7/2}\pi^{3/2}\hbar^3} \exp\left(\frac{\beta^3}{27\beta_0^3}\right) \tag{3.17}$$

Therefore, at low temperature, we have the average energy

$$\varepsilon = -\frac{\partial \operatorname{Log} f(\beta)}{\partial \beta} = -\frac{\beta^2}{9\beta_0^3} \tag{3.18}$$

This confirms that the Gaussian model has a vary large negative average energy at low temperature. It is interesting to note that this result is associated to the diffusion in random media.[2–4]

The Gaussian model was extensively studied previously. The main works in the past were concentrated on the low-energy tail.[9] These results are consistent with our low temperature limit in Eq. (3.18). However, the general partition function in Eq. (3.14) covers the whole temperature range and shows that the behavior of $f(\beta)$ changes as β crosses β_0.

4. FRISCH–LLOYD RANDOM SYSTEMS

The Frisch–Lloyd model[10] has the random potential

$$V(x) = V_0 \sum_{j=1}^{N} \delta(x - x_j) \tag{4.1}$$

where V_0 is fixed and x_j are randomly chosen in $(0, L)$. In the thermodynamic limit, $L \to \infty$, $N \to \infty$, but the density $N/L = n$ remains fixed.

From Eq. (2.8), the average over the random potential can be calculated as follows,

$$\begin{aligned}
&\int_0^L \frac{dx_1}{L} \frac{dx_2}{L} \cdots \frac{dx_N}{L} \exp\left[i \sum_{j=1}^{N} V_0 \psi^2(x_j) \right] \\
&\quad = \left[\frac{1}{L} \int_0^L dx\, e^{iV_0\psi^2(x)} \right]^N = \left[1 - \frac{n}{N} \int_0^L (1 - e^{iV_0\psi^2(x)}) \right]^N \\
&\quad = \exp\left[-n \int_0^L [1 - e^{iV_0\psi^2(x)}]\, dx \right]
\end{aligned} \tag{4.2}$$

This leads to $U(\psi^2) = -in(1 - e^{iV_0\psi^2})$. We introduce $\lambda = 2mV_0/\hbar^2$. From Eq. (2.13), the function ϕ_0 is the solution of the equation

$$\frac{d^2\phi_0}{du^2} - \left[\frac{2mE}{\hbar^2} + \frac{n}{iu} (1 - e^{i\lambda u}) \right] \phi_0 = 0 \tag{4.3}$$

with the boundary condition $\phi_0(0) = 1$ and $\phi_0(\infty) = 0$. At this stage, we have no analytical solution for the above equation, but the partition function can be calculated straightforward.

Let us first apply the general method discussed in Section 2 to Eq. (4.3). Introduce $u = \hbar/\sqrt{2mE}$ and two dimensionless parameters, $\alpha = \lambda/2 = 2mV_0/(n\hbar^2)$ and $\gamma = n\hbar/\sqrt{2mE}$. Then, Eq. (4.3) becomes

$$\frac{d^2\phi_0}{dz^2} = \left[1 + \frac{\gamma}{iz} (1 - e^{i\gamma\alpha z}) \right] \phi_0 \tag{4.4}$$

While γ depends on E, α is independent on E. From Eq. (4.4), we calculate two independent solutions, $\phi_0^{(1)}(z)$ and $\phi_0^{(2)}(z)$ that have the boundary conditions, $\phi_0^{(1)}(0) = 1$, $[d\phi_0^{(1)}/dz]_{|z=0} = 0$, $\phi_0^{(2)}(0) = 0$, and $[d\phi_0^{(1)}/dz]_{|z=0} = 1$, respectively. As both $\phi_0^{(1)}$ and $\phi_0^{(2)}$ have the boundary conditions at $z=0$, they can be integrated easily from Eq. (4.4). The following expression defines a constant g as a function of α and γ,

$$g(\alpha\gamma, \gamma) = -\lim_{z\to\infty} \phi_0^{(1)}(z)/\phi_0^{(2)}(z) \tag{4.5}$$

Then, $\phi_0 = \phi_0^{(1)}(z) + g(\alpha\gamma, \gamma)\,\phi_0^{(2)}(z) \to 0$ as $z\to\infty$ and

$$\mathrm{Re}\left(\frac{d\phi_0}{du}\right)_{|u=0} = \frac{\sqrt{2mE}}{\hbar}\,\mathrm{Re}[g(\alpha\gamma, \gamma)] \tag{4.6}$$

Introduce $\beta_0 = 2m/(n\hbar)^2$, which defines an intrinsic temperature for the model. The one-particle partition function is given by

$$\begin{aligned} f &= -2N\frac{\beta}{\beta_0}\int_0^\infty d\gamma\,\exp\left[-\frac{\beta}{\beta_0\gamma^2}\right]\mathrm{Re}[g(\alpha\gamma, \gamma)]/\gamma^4 \\ &= f(\beta/\beta_0, \alpha\sqrt{\beta/\beta_0}) \end{aligned} \tag{4.7}$$

To examine how the partition function changes with β/β_0, we plot Log(f) in Fig. 2 as a function of β/β_0 and α. As expressed in Eq. (4.7), f is a function of β/β_0 and $\alpha(\beta/\beta_0)^{1/2}$. This provides important scaling information. At high temperature, $\beta \ll \beta_0$, f tends to $\sim\sqrt{\beta_0/\beta}$, leading to the average energy $k_B T/2$. However, at low temperature $\beta \gg \beta_0$, we see that f does not only depend on β/β_0, but also relates to $\alpha\sqrt{\beta/\beta_0}$.

To confirm the above scaling information, we consider the case of small n. At low concentration of impurities, we can expand ϕ_0 in powers of n. Taking

$$\phi_0 = \exp\left(\int_0^u y(s)\,ds\right) \tag{4.8}$$

we then have Eq. (4.3) as

$$y'(u) + y^2(u) = \frac{2mE}{\hbar^2} + \frac{n}{iu}(1 - e^{i\lambda u}) \tag{4.9}$$

The boundary condition $\phi_0(\infty) = 0$ becomes

$$\lim_{u\to\infty}\int_0^u \mathrm{Re}[y(s)]\,ds = -\infty \tag{4.10}$$

In addition,

$$\phi_0'(u)_{|u=0} = y(0) \tag{4.11}$$

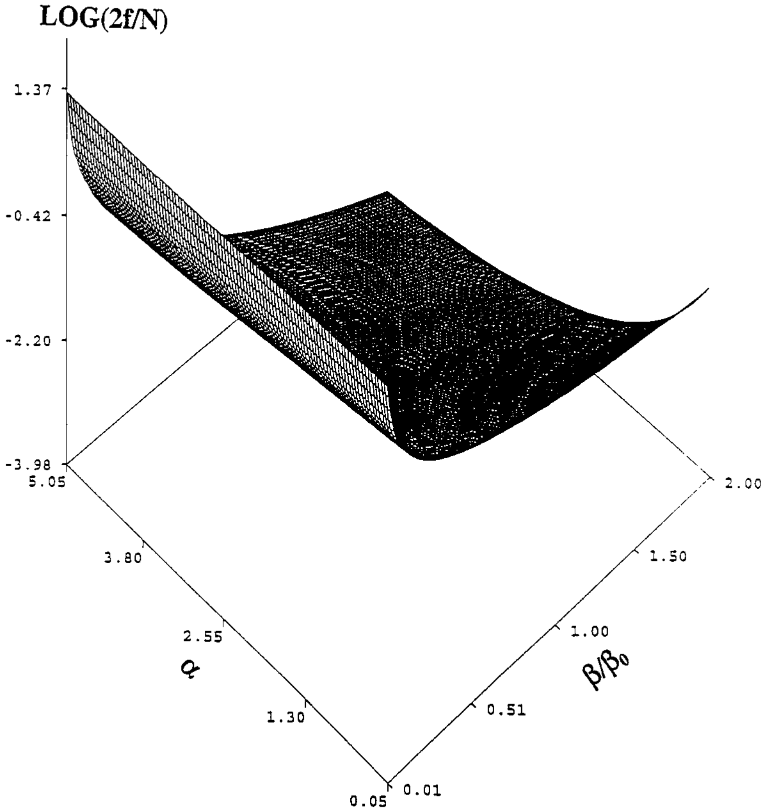

Fig. 2. The partition function $\mathrm{Log}(2f/N)$ as a function of β/β_0 and α for the Frisch–Lloyd model.

As n is small, expanding $y(s)$ in powers of n,

$$y(s) = y_0(s) + ny_1(s) + n^2 y_2(s) + \cdots \tag{4.12}$$

we find

$$\begin{aligned} y_0' + y_0^2 &= \frac{2mE}{\hbar^2} \\ y_1' + 2y_0\, y_1 &= \frac{1 - e^{i\lambda u}}{iu} \\ &\vdots \end{aligned} \tag{4.13}$$

If $E > 0$, we have

$$y_0(u) = -\sqrt{2mE}/\hbar = -k$$
$$y_1(u) = -\int_u^\infty ds\, e^{-k(s-u)}(1 - e^{i\lambda s})/is \tag{4.14}$$
$$\vdots$$

Therefore, if we only keep the first order of n,

$$\mathrm{Re}[\phi_0'(u)_{|u=0}] = -k + n \tan^{-1}(\lambda/2k) \tag{4.15}$$

Similarly, if $E < 0$, we expand ϕ_0 in powers of n and find $\mathrm{Re}[\phi_0'(u)_{|u=0}] = 0$. Thus, the one-particle partition function to the first order of n is given by

$$f = L\sqrt{\frac{m}{2\pi\hbar^2\beta}} + \frac{nL}{2}\left\{1 - \left[1 - \Phi\left(\frac{\lambda\hbar\sqrt{\beta}}{2\sqrt{2m}}\right)\right] e^{\beta\hbar^2\lambda^2/8m}\right\} \tag{4.16}$$

where $\Phi(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2}\, dt$ is the probability function. The result in Eq. (4.16) can be rewritten as

$$f = \frac{N}{2\sqrt{\pi}}\left(\frac{\beta_0}{\beta}\right)^{1/2} + \frac{N}{2}\left\{1 - \left[1 - \Phi\left(\sqrt{\frac{\alpha^2\beta}{4\beta_0}}\right)\right] e^{\alpha^2\beta/4\beta_0}\right\} \tag{4.17}$$

Equation (4.17) is consistent with Eq. (4.7), verifying the scaling information. For example, if at low temperature we have $\beta/\beta_0 > 1$ but $\alpha\sqrt{\beta/\beta_0} < 1$, the average energy from Eq. (4.17) is given by

$$\varepsilon = \left(\frac{k_B T}{2}\right)\frac{(\beta_0/\beta)^{1/2} - \alpha(\beta/\beta_0)^{1/2}}{(\beta_0/\beta)^{1/2} + \alpha(\beta/\beta_0)^{1/2}} \tag{4.18}$$

The Frisch–Lloyd model was widely studied for the Lifshits tail,[11] which corresponds to the low temperature limit here. The current analytical results in Eq. (4.7) and Fig. 2 are well beyond the low temperature limit. The results also identify that the behavior of the partition function changes when β crosses β_0. The important scaling information for the model is a new result.

In summary, the application of the analytical results to the above two models in Sections 3 and 4 shows that the path-integral method is general and powerful. The derived formalism can be readily applicable to many other one-dimensional random systems to investigate the statistical physics of these systems.

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